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Biased Diffusion with Correlated Noise

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The diffusion of hard-core particles subject to a global bias is described by a nonlinear, anisotropic generalization of the diffusion equation with conserved, local noise. Using renormalization group techniques, we analyze the effect of an additional noise term, with spatially long-ranged correlations, on the long-time, long-wavelength behavior of this model. Above an upper critical dimension d_{LR}, the long-ranged noise is always relevant. In contrast, for $d < d_{\mathrm{LR}}$, we find a "weak noise" regime dominated by short-range noise. As the range of the noise correlations increases, an intricate sequence of stability exchanges between different fixed points of the renormalization group occurs. Both smooth and discontinuous crossovers between the associated universality classes are observed, reflected in the scaling exponents. We discuss the necessary techniques in some detail since they are applicable to a much wider range of problems.

KEY WORDS: Nonequilibrium steady states; driven diffusive systems; correlated noise.

I. INTRODUCTION

Systems coupled to multiple energy reservoirs, sustaining net transport currents, are prevalent in nature but fall outside the fundamentally well-understood paradigms of equilibrium statistical mechanics. The characterization and classification of such systems, starting from simple microscopic models, remains a key problem of current research. Since steady states in such systems constitute the closest relatives of equilibrium (Gibbs) states, their study has attracted much recent attention,[(1)] revealing a wealth of unexpected phenomena, even in simple model systems. One of the prime,

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and most elementary, examples of this kind is a system of hard-core particles undergoing biased diffusion. With appropriate boundary conditions, this system, also known as the simple asymmetric exclusion process (ASEP),[2] can exhibit a number of surprising features, including anomalous diffusion,[2–4] long-range spatial correlations,[5] and shocks.[6] One of its key characteristics is the breaking of the usual fluctuation-dissipation theorem.[7]

In the following, we will discuss the effect of long-range correlated noise on this model, formulated via a Langevin equation in continuous space and time. Generated, for example, by correlated fluctuations in the strength of the bias, the additional operator acts in the one-dimensional subspace (labelled "longitudinal") selected by the bias. Its momentum dependence takes the form $q_{\|}^{2(1-\alpha)}$, with $0 \leqslant \alpha \leqslant 1$. The two extreme values, $\alpha = 0$ and $\alpha = 1$, are of particular interest. The choice $\alpha = 0$ corresponds to the standard ASEP. In contrast, $\alpha = 1$ leads to a Langevin equation with *non-conserved* noise, describing biased diffusion of particles which can occasionally be created or annihilated. First introduced by Hwa and Kardar[8] as a continuum description for sliding avalanches in sandpile models, it was analyzed in more detail by Becker and Janssen.[9]

The key advantage of the correlated noise term is that it allows us to incorporate the *whole regime* between these two models, characterized, respectively, by conserved and non-conserved noise. Thus, using field theoretic techniques, we discuss the full renormalization group (RG) flow in (α, d) space, for $0 \leqslant \alpha \leqslant 1$. While the limit $\alpha \to 1$ presents no difficulties, the opposite limit $\alpha \to 0$ is far more subtle: we will see that, just below the upper critical dimension $d_c(\alpha)$, *two* nontrivial renormalization are required to render the $\alpha = 0$ theory finite, in contrast to a *single* one if α is positive and of $O(1)$, leading to an apparent discontinuity in the critical exponents. A similar situation arises in standard ϕ^4-theory with long-ranged interactions.[10] There, however, the discontinuity is entirely spurious and can be removed: letting $\varepsilon = d_c(0) - d$ denote the distance from the upper critical dimension of the short-range theory, the key is to recognize[10] that there is a region of *small* $\alpha = O(\varepsilon)$, where a careful analysis of the RG flow reveals a smooth *connection* between the $\alpha = 0$ and the $\alpha = O(1)$ theories. Thus, all universal scaling properties are shown to depend continuously on α and d. Here, in contrast, the limit $\alpha \to 0$ is even more intricate: while the short-range theory is controlled by two fixed points, only a single one remains in the long-range model. Thus, only one of the two short-range fixed points is smoothly connected to the long-range fixed point, leading to a continuous crossover. The other short-range fixed point eventually loses its stability, accompanied by a discontinuity in critical exponents.

A second feature of our model concerns its close relationship to the KPZ[11] or Burgers[12] equation with correlated noise.[13, 14] It is well known that the continuum description for the ASEP and the noisy Burgers equation coincide in one spatial dimension. In contrast to the Burgers equation, the ε-expansion for biased diffusion around $d_c = 2$ presents no difficulties, and numerous results can be obtained exactly, to all orders in perturbation theory. These may therefore be analytically continued to $d = 1$ and are expected to hold for both the ASEP and the KPZ equation. In fact, to the extent that exact solutions[15] are available, this expectation is indeed confirmed. Thus, our analysis bears some relation to the behavior of the one-dimensional Burgers equation with correlated noise.

This paper is organized as follows. We first introduce the Langevin equation for the "short-range" (SR) version of our model, corresponding to the usual d-dimensional ASEP with translational invariance. We then add a second noise term with long-range spatial correlations and propose two possible microscopic mechanisms for such a term. Turning to the RG analysis, we first summarize the SR case, $\alpha = 0$. The "long-range" (LR) theory associated with $\alpha = O(1)$ is presented next, and the limit $\alpha \to 1$ is discussed. Finally, we introduce the "hybrid" model, characterized by $\alpha = O(\varepsilon)$, which interpolates between the SR ($\alpha = 0$) and LR ($\alpha = O(1)$) theories. Its RG flow is computed in a double expansion, where both α and ε are small parameters. The different fixed points are interpreted and their stability is evaluated, illustrating how the subtle interpolation from $\alpha = 0$ to $\alpha = O(1)$ occurs. We conclude with some comments and open questions.

II. THE MODEL

We consider a d-dimensional system of hard-core particles, which are allowed to diffuse on a regular lattice with fully periodic boundary conditions. The bias, reminiscent of an electric field $\mathbf{E} = E\mathbf{e}_{||}$ acting on charges, favors particle moves along a specific ("longitudinal") direction along unit vector $\mathbf{e}_{||}$. The long-time, long-wavelength properties of this model are most conveniently captured by a coarse-grained description in continuous space and time. Here, only a single slow variable is needed, namely, the local particle density $c(\mathbf{r}, t)$. Since the associated Langevin equation has been discussed previously,[3, 4] we describe it only briefly here. Due to particle number conservation, it takes the form of a continuity equation, $\partial_t c + \nabla \mathbf{j} = 0$. In addition to a diffusive term, the current $\mathbf{j}$ contains an Ohmic contribution, $\mathbf{j}_E = \kappa(c)\,\mathbf{E}$, induced by the bias, and a Gaussian noise termed modelling the fast degrees of freedom. By virtue of the excluded volume constraint, the conductivity $\kappa(c)$ must vanish with both particle and hole density, i.e., $\kappa(c) \propto c(1-c)$. Aiming for a perturbation expansion around a field with zero average, we introduce the deviations $s(\mathbf{r}, t) = c(\mathbf{r}, t) - \bar{c}$ from the average density $\bar{c}$. A term linear in s, arising from the expansion of $\kappa(c)$, can be absorbed via a Galilei transformation. Finally, we allow for anisotropies in the diffusion coefficient and the correlations of the Langevin noise, since the bias singles out a preferred direction. Neglecting contributions that are irrelevant in the long-wavelength, long-time limit, the Langevin equation can be summarized in the form:[4]

$$\partial_t s = \lambda \left\{ (\nabla_\perp^2 + \rho \partial^2)\, s + \frac{g}{2} \partial s^2 \right\} + \eta(\vec{r}, t) \tag{1}$$

Here, $\nabla_\perp(\partial)$ denote the gradients in the transverse (longitudinal) subspace, and the kinetic coefficient λ defines a time scale. The coupling $g \propto E$ captures the effects of the drive. The noise $\eta(\mathbf{r}, t)$ is Gaussian so that two moments suffice to characterize its full distribution (after a suitable resealing of the variables):

$$\begin{aligned} \langle \eta(\mathbf{r}, t) \rangle &= 0 \\ \langle \eta(\mathbf{r}, t)\, \eta(\mathbf{r}', t') \rangle &= -2\lambda (\nabla_\perp^2 + \sigma \partial^2)\, \delta(\mathbf{r} - \mathbf{r}')\, \delta(t - t') \end{aligned} \tag{2}$$

The differential operator $(\nabla_\perp^2 + \sigma \partial^2)$ in the second moment ensures that the conservation law is strictly obeyed. Clearly, the correlations described by this form are purely local, so that we will refer to Eqs. (1) and (2) as the "short-range" (SR) theory. Its universal properties have been discussed in refs. 4 and 5.

The simplest way of introducing a correlated noise into this equation is to *add* a long-range correlated noise to the local one. Thus, Eq. (2) is amended to

$$\begin{aligned} \langle \eta(\mathbf{r}, t) \rangle &= 0 \\ \langle \eta(\mathbf{r}, t)\, \eta(\mathbf{r}', t') \rangle &= 2\lambda [-(\nabla_\perp^2 + \sigma \partial^2) + b(-\partial^2)^{1-\alpha}]\, \delta(\mathbf{r} - \mathbf{r}')\, \delta(t - t') \end{aligned} \tag{3}$$

The new operator $b(-\partial^2)^{1-\alpha}$ reflects a power-law decay of the noise correlations in real space, giving rise to $a = q_{\|}^{2(1-\alpha)}$ momentum dependence in Fourier space. Clearly, setting α to zero reproduces the SR theory, albeit with σ replaced by $\sigma + b$. In contrast, choosing $\alpha = 1$ generates a non-conserved noise, playing the role of a random source term in Eq. (1). Note that the vanishing of the first moment ensures that the particle density, while not strictly invariant under the dynamics, remains conserved *on average*. The long-time, long-wavelength behavior of this theory has been analyzed in refs. 8 and 9. Other choices of long-ranged noise correlations are of course possible.[16] The version adopted above is "minimal" in the sense that, first, it leads to a nontrivial, renormalizable theory, and second, it allows us to discuss the full regime between the conserved ($\alpha=0$) and non-conserved ($\alpha=1$) cases using just one additional operator. Microscopically such long-ranged contributions to the Langeyin noise can be generated in two ways. One option is to impose a bias, along the $\mathbf{e}_{||}$ direction, which consists of a spatially uniform (E_o) and a locally random component, $E_r(\mathbf{r}, t)\, \mathbf{e}_{||}$. The latter is controlled by a Gaussian distribution with long-range correlations: $\langle E_r(\mathbf{r}, t)\, E_r(\mathbf{r}', t') \rangle = 2\lambda b C(\mathbf{r}-\mathbf{r}')\, \delta(t-t')$, decaying at large distances according to $C(\mathbf{x}) \sim 2\lambda b\, |\mathbf{x}|^{2\alpha-d}$. Such a distribution is easily realized in simulations. Adding the random component to the Ohmic current generates an additional term, $\partial \bar{c}(1-\bar{c})\, E_r(\mathbf{r}, t)$, on the right hand side of (1). Averages over the random variable E_r can now be treated completely analogously to those over the thermal noise, (2). A second option is to retain the strictly uniform bias but randomly add or remove particles, according to a long-ranged distribution. More details will be presented in ref. 16.

Due to the presence of the nonlinearity, the computation of averages over the noise distribution requires a perturbative approach, coupled with renormalization group methods. For these purposes, it is most convenient to recast the Langevin equations as dynamic functionals.[17] Introducing a Martin–Siggia–Rose[18] response field $\tilde{s}(\mathbf{r}, t)$, we can reexpress Eqs. (1) and (3) in the form

$$\mathscr{J}[\tilde{s}, s] = \int dt \int d^d r \left\{ \tilde{s}\partial_t s - \lambda \tilde{s} \left[(\nabla_\perp^2 + \rho\partial^2)\, s + \frac{g}{2}\, \partial s^2 \right] \right. \\ \left. + \lambda \tilde{s} [(\nabla_\perp^2 + \sigma\partial^2) - b(-\partial^2)^{1-\alpha}]\, \tilde{s} \right\} \tag{4}$$

so that correlation and response functions can be computed as functional averages with weight $\exp(-\mathscr{J})$. We stress that the SR model is recovered from this general expression, simply by setting $b=0$.

As a first step towards the RG analysis of this model, we discuss its scaling symmetries. First, under a global rescaling of all coordinates, $\vec{r} \to \mu^{-1}\vec{r}$, we find a scale invariant theory (as long as we neglect any cut-off-dependence) provided $\lambda t \to \mu^{-2}\lambda t$, $s \to \mu^{(d-2\alpha)/2} s$ and $\tilde{s} \to \mu^{(d+2\alpha)/2}\tilde{s}$. Moreover, we obtain $g \to \mu^{(1+\alpha)-d/2} g$ which allows us to identify the upper critical dimension $d_c(\alpha) = 2(1+\alpha)$. To avoid confusion, we will reserve the symbol ε for the SR case, i.e., $\varepsilon \equiv 2-d$. For the general case, we define $\bar{\varepsilon} \equiv d_c(\alpha) - d = 2(1+\alpha) - d$. Second, we may rescale the parallel coordinate alone, due to the anisotropy of the model: under $x_{||} \to \beta x_{||}$, scale invariance is retained if $s \to \beta^{-1/2} s$, $\tilde{s} \to \beta^{-1/2} s$, $\rho \to \beta^2 \rho$, $\sigma \to \beta^2 \sigma$, $b \to \beta^{2(1-\alpha)} b$ and $g \to \beta^{3/2} g$. This permits us to identify a set of *effective* couplings, namely, $u = G_d g^2 \rho^{-5/2} \sigma$, $w = \rho/\sigma$ and $v = b\rho^{\alpha}/\sigma$. These will emerge quite naturally in perturbation theory, combined with appropriate powers of μ which absorb the remaining momentum dependence. For convenience, a geometric factor $G_d = S_d/(2\pi)^d$, with S_d the surface of the d-dimensional unit sphere, has been absorbed into u. For later reference, we note that only the positive octant $u \geqslant 0$, $w \geqslant 0$ and $v \geqslant 0$ corresponds to physical theories even though the RG flow can be discussed in a larger parameter space. Finally, the dynamic functional (4) exhibits a nontrivial continuous symmetry, with parameter a, which is characterized by the "Galilean" transformation $s(\mathbf{r}, t) \to s(\mathbf{r} + \lambda a g \mathbf{e}_{||} t, t) + a$, $\tilde{s}(\mathbf{r}, t) \to \tilde{s}(\mathbf{r} + \lambda a g \mathbf{e}_{||} t, t)$.

So far, we were able to consider the case of general $0 \leqslant \alpha \leqslant 1$. The next section will show, however, that this procedure cannot be continued for the actual perturbative calculation of correlation and response functions.

III. RENORMALIZATION GROUP ANALYSIS

We now turn to perturbation theory, with the goal of identifying those correlation and response functions that require renormalization, due to the presence of ultraviolet (UV) divergences in Feynman diagrams. As usual, we focus on the one-particle irreducible (1PI) vertex functions with $\tilde{n}$ (n) external $\tilde{s}$- (s-) legs, $\Gamma_{\tilde{n}n}$. These will be computed in dimensional regularization using minimal subtraction. Since the models associated with $\alpha = 0$ and $\alpha = 1$ have been discussed previously (in refs. 4, 5, 8, and 9, respectively), we only briefly summarize their RG structure, focusing predominantly on the theory with $0 < \alpha < 1$.

A. The Short-Range Theory: $\alpha = 0$

We first review the model with $\alpha = 0$.[4, 5] Here, the coupling b and the effective v are redundant and will be set to zero. The upper critical dimension is $d_c(0) = 2$. Straightforward dimensional analysis shows that there are three naively divergent vertex functions: Γ_{11}, Γ_{20} and Γ_{12}. A Ward identity based on the "Galilean" symmetry shows, however, that Γ_{12} requires no additional renormalization once Γ_{11} and Γ_{20} have been rendered finite. Letting the superscript $^\circ$ denote bare couplings, the renormalized couplings are defined in the usual way: $\mathring{\rho} = Z_\rho^{\mathrm{SR}} \rho$, $\mathring{\sigma} = Z_\rho^{\mathrm{SR}} \sigma$, $\mathring{g} = Z_g g \mu^{\varepsilon/2}$, where $\varepsilon = 2 - d$. The parameter μ sets a typical momentum scale which will control the RG flow. All SR functions are marked with an explicit superscript to distinguish them from their finite α counterparts. The UV divergences in Γ_{11} and Γ_{20} give rise to nontrivial Z-factors for ρ and σ, which can be computed perturbatively in u. A one-loop calculation yields $Z_{\rho}^{\mathrm{SR}} = u(3w+1)/(16\varepsilon) + O(u^2)$ and $Z_{\sigma}^{\mathrm{SR}} = u(3w^2 + 2w+3)/(32\varepsilon) + O(u^2)$. We emphasize that w is *not* treated perturbatively here; therefore, $O(1)$ fixed points for w should not come as a surprise later. Finally, $Z_g = 1$ to all orders in u, due to the "Galilean" invariance. To complete the discussion, we introduce the Wilson functions

$$
\begin{aligned}
\zeta_{\rho}^{\mathrm{SR}} &\equiv \mu\partial_{\mu} \ln \rho \,|_{bare} = -u\,\frac{3w+1}{16} + O(u^2) \\
\zeta_{\sigma}^{\mathrm{SR}} &\equiv \mu\partial_{\mu} \ln \rho \,|_{bare} = -u\,\frac{3w+{}^2+2w+3}{32} + O(u^2)
\end{aligned}
\tag{5}
$$

The RG flow is expressed through the μ-dependence of the renormalized couplings $u = G_d g^2 \rho^{-5/2}\sigma$ and $w = \rho/\sigma$, especially in the scaling limit $\mu \to 0$:

$$
\begin{aligned}
\beta_{u}^{\mathrm{SR}}(u,w) &\equiv \mu\partial_{\mu} u \,|_{bare} = -\left[\varepsilon + \tfrac{5}{2}\zeta_{\rho}^{\mathrm{SR}} - \zeta_{\sigma}^{\mathrm{SR}}\right] u \\
\beta_{w}^{\mathrm{SR}}(u,w) &\equiv \mu\partial_{\mu} w \,|_{bare} = \left[\zeta_{\rho}^{\mathrm{SR}} - \zeta_{\sigma}^{\mathrm{SR}}\right] w
\end{aligned}
\tag{6}
$$

In this form, the right hand sides of Eq. (6) are exact to all orders.

These flow equations possess three fixed points, marked by the vanishing of both β_{u}^{SR} and β_{w}^{SR}: (i) $u^* = 8\varepsilon/3 + O(\varepsilon^2)$, $w^* = 1$; (ii) $u^* = 16\varepsilon + O(\varepsilon^2)$, $w^* = 0$; (iii) $u^* = 16\varepsilon/3 + O(\varepsilon^2)$, $w^* = 1/3 + O(\varepsilon)$; and finally a fixed line (iv) $u^* = 0$, with arbitrary w. The stability of these fixed points (and line) can be expressed through the 2×2 matrix $\mathbb{M}_{\mathrm{SR}} \equiv (\partial_i \beta_j^{\mathrm{SR}})^*$, $i, j = u, v$, of derivatives of the β-functions, evaluated at the fixed points. We find that the fixed line (iv) is stable only for $\varepsilon < 0$, i.e., $d > 2$, and will therefore be labelled a "Gaussian" line. Focusing on $\varepsilon > 0$, we find that (i) and (ii) are both locally stable. Fixed point (iii) is hyperbolic and sits on the separatrix $w = 1/3 + O(\varepsilon)$ which separates the domains of attraction of (i) and (ii). Flow lines starting in the region $w > 1/3 + O(\varepsilon)$ are attracted towards fixed point (i), and vice versa. We refer to fixed point (i) as the FDT-restoring short-range fixed point, since theories with $w = 1$ can be shown to possess a higher degree of symmetry, associated with satisfying the FDT.[4] Here, Eq. (6) allows us to compute the fixed point values of $\zeta_{\rho}^{\mathrm{SR}}$ and $\zeta_{\sigma}^{\mathrm{SR}}$ exactly, to all orders in ε: $\zeta_{\rho}^{\mathrm{SR}*} = \zeta_{\sigma}^{\mathrm{SR}*} = -2\varepsilon/3$. Conversely, fixed point (ii) violates the FDT, and $\zeta_{\rho}^{\mathrm{SR}*} = -\varepsilon + O(\varepsilon^2)$; $\zeta_{\sigma}^{\mathrm{SR}*} = -3\varepsilon/2 + O(\varepsilon^2)$ are known only perturbatively.

Finally, we comment briefly on the scaling properties associated with these fixed points,[4, 5] which arise from the RG equations for they vertex functions. For example, the dynamic density correlation function (structure factor) $S(\mathbf{q}, \omega) \equiv \langle s(-\mathbf{q}, -\omega)\, s(\mathbf{q}, \omega) \rangle$ scales as

$$S(\mathbf{q}, \omega) = l^{-2} S(q_{||} l^{-1-\Delta_{\mathrm{SR}}}, q_{\perp} l^{-1}, \omega l^{-2}) \tag{7}$$

This reflects the emergence of anomalous diffusion, characterized, near fixed point (i), by a strong anisotropy exponent[1] $\Delta_{\mathrm{SR}} \equiv -\zeta_{\rho}^{\mathrm{SR}*}/2 = (2-d)/3$, and, near fixed point (ii) by an exponent $\Delta_{\mathrm{SR}} \equiv -\zeta_{\rho}^{\mathrm{SR}*}/2 = \varepsilon/2 + O(\varepsilon^2)$. Next, we turn to the case of finite α.

B. The Long-Range Models with $\alpha = O(1)$

As indicated in the Introduction, we will have to distinguish between values of α which are $O(2-d)$ versus those which are $O(1)$. Here, we analyze the second case, which corresponds to true long-range (LR) theories. The first case, being a "hybrid" between short-range and long-range models, will be deferred to the next subsection.

Recalling our discussion in Sect. 2, the critical dimension is now $d_c(\alpha) = 2(1+\alpha)$, and we define $\bar{\varepsilon} = 2(1+\alpha) - d$, to be distinguished from $\varepsilon = 2-d$. The operator $\tilde{s}(\nabla_{\perp}^2 + \sigma \partial^2)\, \tilde{s}$ is clearly irrelevant compared to $b\tilde{s}(-\partial^2)^{1-\alpha}\, \tilde{s}$ and may be dropped. The naive dimensions of the fields s and $\tilde{s}$ are α-dependent, and as a result only *two* vertex functions, namely Γ_{11} and Γ_{12}, are naively divergent. In contrast to the short-range case Γ_{20} is naively convergent here. Moreover, all divergent contributions to any vertex function are polynomial in the momenta,[10] so that b is not renormalized. The "Galilean" invariance still holds, so that Γ_{12}, and hence g, requires no new Z-factor. Thus, defining renormalized couplings according to $\mathring{\rho} = Z_{\rho}^{\mathrm{LR}}$, $\mathring{b} = Z_b b$, $\mathring{g} = Z_g g \mu^{\bar{\varepsilon}/2}$, we find $Z_b = Z_g = 1$ to all orders, and $Z_{\rho}^{\mathrm{LR}} = 1 - \bar{u} A(\alpha)/\bar{\varepsilon} + O(\bar{u}^2)$, to first order in $\bar{u} = uv$. The latter is the appropriate effective coupling here, since σ no longer appears in the theory. The coefficient $A(\alpha) = \Gamma(1+\alpha)\, \Gamma(3/2 - \alpha)(1+2\alpha)/(8\sqrt{\pi})$ controls the one-loop divergence in Γ_{11}. The Wilson functions are easily obtained:

$$\zeta_{\rho}^{\mathrm{LR}} \equiv \mu \partial_{\mu} \ln \rho\,|_{bare} = -\bar{u} A(\alpha) + O(\bar{u}^2) \tag{8}$$

and

$$\beta_{\bar{u}}^{\mathrm{LR}} \equiv \mu \partial_{\mu} \bar{u}\,|_{bare} = -\left[\bar{\varepsilon} + \left(\tfrac{5}{2} - \alpha\right) \zeta_{\rho}^{\mathrm{LR}}\right] \bar{u} \tag{9}$$

Clearly, the LR theory exhibits one (Gaussian) fixed point $\bar{u}^* = 0$ which is easily shown to be stable above the upper critical dimension, and a non-trivial fixed point $\bar{u}^* = 2\bar{\varepsilon}/[(5-2\alpha)\,A(\alpha)] + O(\bar{\varepsilon}^2)$, stable below d_c. At this fixed point, the value of $\zeta_\rho^{\mathbf{LR}}$ is again known to all orders in $\bar{\varepsilon}$, namely, $\zeta_\rho^{\mathbf{SR}*} = -2\bar{\varepsilon}/(5-2\alpha)$.

The scaling properties of the long-range models are again easily derived from an RG equation for the vertex functions. For comparison with the SR case, we quote the result for the dynamic structure factor:

$$S(\mathbf{q}, \omega) = l^{-2(1+\alpha)} S(q_{||} l^{-1-\Delta_{\mathrm{LR}}}, q_\perp l^{-1}, \omega l^{-2}) \tag{10}$$

Once again, we observe anomalous diffusion; here, however, it is controlled by the LR exponent $\Delta_{\mathbf{LR}} \equiv \zeta_\rho^{\mathbf{SR}*}/2 = (7-d)/(5-2\alpha) - 1$.

It is obvious at this point that the limit $\alpha \to 0$, taken in the Wilson function (8) or the correlation function (10), will not restore the SR theory First, $\lim_{\alpha \to 0} \zeta_\rho^{\mathbf{LR}} = -\bar{u}/16 + O(\bar{u}^2)$ which reproduces only the $w = 0$ component of $\zeta_\rho^{\mathbf{SR}}$. Second, this naive limit cannot generate a non-vanishing $\zeta_\sigma^{\mathbf{SR}}$ which plays a key role in the SR theory. Thus, not surprisingly, we do not observe a smooth crossover from the SR to the LR theory, by simply letting α tend to zero in this naive fashion. An elegant way of dealing with this difficulty was suggested in ref. 10: By treating α as a small parameter of $O(\varepsilon)$ and expanding in both ε and α, one can "magnify" the small α-section of (d, α) space and resolve the crossover between the SR and LR fixed points. Concerning the scaling form of the structure factor, the prefactor $l^{-2(1+\alpha)}$ originates entirely in the bare dimensions of the fields. Since these are well defined under a naive $\alpha \to 0$ limit (cf. Sect. 2) and remain unrenormalized, we have $\lim_{\alpha \to 0} l^{-2(1+\alpha)} = l^{-2}$ without further complications. The exponent $\Delta_{\mathbf{LR}}$, however, must be tracked much more carefully.

Before turning to these subtleties, we briefly consider the other limit $\alpha \to 1$. First, we summarize the $\alpha = 1$ theory,[9] with $\bar{\varepsilon} = 4 - d$, in its own right. Here, b measures the strength of the (non-conserved) noise. The effective coupling is $\bar{u} = G_d g^2 \rho^{-3/2} b$. According to ref. 9, there is only one nontrivial Z-factor, namely $Z_\rho = 1 - 3\bar{u}/(8\varepsilon) + O(\bar{u}^2)$. Again, $Z_b = Z_u = 1$ to all orders. Thus, the theories with $\alpha = 1$ and $0 < \alpha < 1$ exhibit the same set of UV divergences. From the preceding discussion, it is therefore clear that we may anticipate a smooth crossover here. This is explicitly confirmed by the results of ref. 9 for $\alpha = 1$: two fixed points $\bar{u}^* = 0$ and $\bar{u}^* = 16\varepsilon/9 + O(\bar{\varepsilon}^2)$, are found, stable above and below $d_c = 4$, respectively. At the nontrivial fixed point, $\Delta = -\zeta_\rho^*/2$ takes the exact value $\varepsilon/3$. It is easy to check that this agrees with the $\alpha \to 1$ limit of Eqs. (8)–(10), so that this limit is continuous.

C. The Hybrid Theory: $\alpha = O(\epsilon)$

Finally, we turn to the analysis of the key region in (d, α) space, namely, $\alpha = O(\varepsilon)$. The naive $\alpha \to 0$ limit presupposes $\varepsilon \ll \alpha$, and hence fails to resolve the crossover between the SR and LR theories which occurs for $\alpha = O(\varepsilon)$. With both α and ε small, we follow ref. 10 and analyze the general dynamic functional, Eq. (4), near the upper critical dimension of the SR theory, i.e., $d = 2$. Both couplings σ and b will be retained, and their RG flow will be studied. We refer to this model as the "hybrid" theory, since it contains the vital elements of both SR and LR cases. Thus, the hybrid theory will exhibit a well-defined $\alpha \to 0$ limit. Once again, we will be able to obtain a number of results to all orders in $\varepsilon = 2 - d$.

Our first task will be to identify a set of suitable couplings which allow us to interpolate between the SR and the LR theories. Guided by ref. 10, we consider the general structure of the Wilson functions in the SR and the hybrid model. The appropriate effective couplings will be those that map those functions onto one another.

To obtain a general form for the Wilson functions, we begin with the Z-factors and construct them order by order in perturbation theory. Focusing on the SR theory first, a typical graph of Γ_{11}, at L-loop, contains $2L$ vertices, L (bare) correlators and $2L - 1$ (bare) propagators. Each vertex carries a factor of $\mathring{g} q_{||}$, and each correlator contributes a factor $(q_{\perp}^2 + \sigma q_{||}^2)$, where $(\omega; \mathbf{q}) = (\omega; q_{||}, \mathbf{q}_{\perp})$ denote the frequency and momentum of the corresponding line. Symmetry requires that 2, out of the L parallel momentum factors contributed by the vertices, will not be integrated over. Moreover, all denominators are generically of the form $(i\omega\lambda^{-1} + q_{\perp}^2 + \rho q_{||}^2)$. Changing the parallel integration variable from $q_{||}$ to $\rho^{1/2} q_{||}$, it is straightforward to show that such a graph will give rise to a contribution of the form $u^L \sum_{m=0}^{L} w^m A_{Lm}/(L\varepsilon) + O(\varepsilon^{-2})$ in Z_{ρ}^{SR}. Here, the A_{Lm} are the numerical coefficients of the $O(\varepsilon^{-1})$ poles, in minimal subtraction. Higher order poles appear also, of course, but cancel in the Wilson functions and need not be considered for this reason. Generic L-loop graphs of Γ_{20} have a similar structure, with $L + 1$ correlators and $2L - 2$ propagators, and the coefficients of the lowest order ε-poles will be denoted by B_{Lm} here. Collecting, we obtain the Z-factors in the general form:

$$
\begin{aligned}
Z_{\rho}^{\mathrm{SR}} &= 1 + \sum_{L=1}^{\infty} u^L \sum_{m=0}^{L} w^m \frac{A_{Lm}}{L\varepsilon} + O(\varepsilon^{-2}) \\
Z_{\sigma}^{\mathrm{SR}} &= 1 + \sum_{L=1}^{\infty} u^L \sum_{m=0}^{L+1} w^m \frac{B_{Lm}}{L\varepsilon} + O(\varepsilon^{-2})
\end{aligned}
\tag{11}
$$

The general form of the Wilson functions is easily found from the above:

$$\zeta_\rho^{SR} = \sum_{L=1}^{\infty} u^l \sum_{m=0}^{L} w^m A_{Lm}$$
$$\zeta_\sigma^{SR} = \sum_{L=1}^{\infty} u^l \sum_{m=0}^{L+1} w^m B_{Lm} \tag{12}$$

Next, we consider the hybrid case. Two key differences emerge: First, each correlator now contributes a factor of $(q_\perp^2 + \sigma q_{||}^2 + b q_{||}^{2(1-\alpha)})$ to a given diagram. This product gives rise to a sum of individual terms each of which contains $L-n$ factors of type $(q_\perp^2 + \sigma q_{||}^2)$ and n factors of type $b q_{||}^{2(1-\alpha)}$, with $n = 0, 1,..., L$. Second, since α is now of $O(\varepsilon)$, the poles in these individual contributions are proportional to $1/(L\varepsilon + 2n\alpha)$.[10] Following the same reasoning as above, and recalling the effective coupling $v = b\rho^\alpha/\sigma$, one finds quite readily that

$$Z_\rho = 1 + \sum_{L=1}^{\infty} u^L \sum_{n=0}^{L} \sum_{m=0}^{L-n} \binom{L}{n} w^m v^n \frac{A_{Lmn}}{L\varepsilon + 2n\alpha} + O(\varepsilon^{-2})$$
$$Z_\sigma = 1 + \sum_{L=1}^{\infty} u^L \sum_{n=0}^{L+1} \sum_{m=0}^{L+1-n} \binom{L+1}{n} w^m v^n \frac{B_{Lmn}}{L\varepsilon + 2n\alpha} + O(\varepsilon^{-2}) \tag{13}$$

Here, A_{Lmn} and B_{Lmn} are the numerical coefficients of each pole. In minimal subtraction, they are independent of both ε and α, since the latter is also $O(\varepsilon)$ here. Clearly, the associated Wilson functions take the form

$$\zeta_\rho = \sum_{L=1}^{\infty} u^L \sum_{n=0}^{L} \sum_{m=0}^{L-n} \binom{L}{n} w^m v^n A_{Lmn}$$
$$\zeta_\sigma = \sum_{L=1}^{\infty} u^L \sum_{n=0}^{L+1} \sum_{m=0}^{L+1-n} \binom{L+1}{n} w^m v^n B_{Lmn} \tag{14}$$

Next, we establish a relationship between the coefficients (A_{Lm}, B_{Lm}) and (A_{Lmn}, B_{Lmn}), by considering the $\alpha \to 0$ limit of the hybrid theory. In this limit, the dynamic functional, Eq. (4), simply reduces to that of the SR theory, with σ replaced by $\sigma + b$. Therefore, we can relate $\lim_{\alpha\to 0} Z_\rho$ and $\lim_{\alpha\to 0} Z_\sigma$ to Z_ρ^{SR} and Z_σ^{SR}. Due to the replacement of σ by $\sigma + b$, the SR Z-factors depend on a modified set of couplings. Defining

$$\bar{u} \equiv u(1+v), \qquad \bar{w} \equiv \frac{w}{1+v}$$

we obtain:

$$\lim_{\alpha \to 0} Z_\rho(u, w, v) = Z_\rho^{\mathrm{SR}}(\bar{u}, \bar{w})$$
$$\lim_{\alpha \to 0} Z_\sigma(u, w, v) = (1+v)\, Z_\sigma^{\mathrm{SR}}(\bar{u}, \bar{w}) - v \tag{15}$$

Inserting the explicit forms for the Z-factors into Eq. (15) and recalling that (A_{Lmn}, B_{Lmn}) are independent of α, we read off the relation between the two sets of coefficients:

$$\binom{L}{n} A_{Lmn} = \binom{L-m}{n} A_{Lm}$$
$$\binom{L}{n} B_{Lmn} = \binom{L+1-m}{n} B_{Lm} \tag{16}$$

Combining Eqs. (16) and (14) provides us with an *identity* between SR and hybrid Wilson functions:

$$\zeta_\rho(u, w, v) = \zeta_\rho^{\mathrm{SR}}(\bar{u}, \bar{w})$$
$$\zeta_\sigma(u, w, v) = (1+v)\, \zeta_\sigma^{\mathrm{SR}}(\bar{u}, \bar{w}) \tag{17}$$

Here, ζ_ρ^{SR} and $\zeta_\sigma^{\mathrm{SR}}$ denote the Wilson functions of the SR model, Eq. (5), evaluated at $(\bar{u}, \bar{w})$. In the following, we use the abbreviated notation $\bar{\zeta}_\rho \equiv \zeta_\rho^{\mathrm{SR}}(\bar{u}, \bar{w})$ and $\bar{\zeta}_\sigma \equiv \zeta_\sigma^{\mathrm{SR}}(\bar{u}, \bar{w})$.

Equation (17) form the basis for the remainder of the paper: they hold the key for the discussion of the hybrid theory and for the desired matching between LR and SR models. It is particularly gratifying that they are valid *to all orders* in perturbation theory. An explicit calculation of the hybrid Wilson functions, instead of the general considerations presented above, would establish Eq. (17) only up to a given order.

To discuss the RG flow for the hybrid theory, we compute the β-functions for the couplings $(\bar{u}, \bar{w}, v)$:

$$\beta_{\bar{u}}(\bar{u}, \bar{w}, v) = \mu \partial_\mu \bar{u}\,|_{bare} = -\left[\varepsilon + \frac{5}{2}\bar{\zeta}_\rho - \bar{\zeta}_\sigma + \alpha \frac{v}{1+v}(2 - \bar{\zeta}_\rho)\right] \bar{u}$$
$$\beta_{\bar{w}}(\bar{u}, \bar{w}, v) = \mu \partial_\mu \bar{w}\,|_{bare} = \left[\bar{\zeta}_\rho - \bar{\zeta}_\sigma + \alpha \frac{v}{1+v}(2 - \bar{\zeta}_\rho)\right] \bar{w} \tag{18}$$
$$\beta_v(\bar{u}, \bar{w}, v) = \mu \partial_\mu v\,|_{bare} = -[(1+v)\, \bar{\zeta}_\sigma + \alpha(2 - \bar{\zeta}_\rho)]\, v$$

and seek their fixed points $(\bar{u}^*, \bar{w}^*, v^*)$. These fall into two groups.

The *first* group is characterized by $v^* = 0$: In this case, Eqs. (18) reduce to the fixed point equations (6) for the SR theory, so that we recover the equivalents of the familiar SR fixed points, namely (i) the FDT satisfying $(8\varepsilon/3 + O(\varepsilon^2), 1, 0)$, (ii) the FDT violating $(16\varepsilon + O(\varepsilon^2), 0, 0)$, (iii) the fixed point $(16\varepsilon/3 + O(\varepsilon^2), 1/3 + O(\varepsilon), 0)$ on the separatrix between (i) and (ii), and finally the Gaussian fixed line (iv) $(0, \bar{w}, 0)$. In contrast to the SR case, however, their stability needs to be investigated in the *three*-dimensional space spanned by $(\bar{u}, \bar{w}, v)$, controlled by the 3×3 matrix $\mathbb{M} \equiv (\partial_i \beta_j)^*$, $i, j = \bar{u}, \bar{w}, v$. For $v^* = 0$, we find $\partial_{\bar{u}} \beta_v = \partial_{\bar{w}} \beta_v = 0$ at all of these fixed points, so that only the upper left 2×2 corner of $\mathbb{M}$ (which is just $\mathbb{M}^{SR}$) and the bottom right element, $\partial_v \beta_v = -\bar{\zeta}_\sigma - \alpha(2 - \bar{\zeta}_\rho)$, are relevant. The eigenvalues of $\mathbb{M}^{SR}$ were already computed in Sect. (3.1), so that only fixed points (i) and (ii) remain as candidates for global stability if $\varepsilon > 0$. For fixed point (i), $\bar{\zeta}_\rho^* = \bar{\zeta}_\sigma^* = -2\varepsilon/3$ to all orders (cf. Sect. 3.1) so that $\partial_v \beta_v$ is positive for $\alpha < \alpha_1 \equiv \varepsilon/(3 + \varepsilon)$. Thus, α_1 demarcates the stability boundary of the FDT satisfying SR fixed point (i). Considering fixed point (ii), we have $\bar{\zeta}_\rho^* = -\varepsilon + O(\varepsilon^2)$ and $\bar{\zeta}_\sigma^* = -3\varepsilon/2 + O(\varepsilon^2)$, giving positive values for $\partial_v \beta_v$, provided $\alpha < \alpha_2 \equiv 3\varepsilon/4 + O(\varepsilon^2)$. So, the FDT violating SR fixed point (ii) remains stable for a larger region of a than its FDT-satisfying partner. We remark that, in contrast to α_1, α_2 is known only perturbatively. Finally, the fixed line $(0, \bar{w}, 0)$ is stable provided both ε and α are negative.

Returning to $\varepsilon > 0$, it is clear that another fixed point must become stable beyond α_2. By necessity, this can only be a member of the *second* group, having $v^* \neq 0$. For all of these, the $[\ \cdots\ ]$ bracket in the last line of Eqs. (18) vanishes, so that the first two equations simplify to

$$\begin{aligned} \beta_{\bar{u}}(\bar{u}, \bar{w}, v^*) &= -[(\varepsilon + 2\alpha) + (\tfrac{5}{2} - \alpha)\, \bar{\zeta}_\rho]\, \bar{u} \\ \beta_{\bar{w}}(\bar{u}, \bar{w}, v^*) &= [2\alpha + (1 - \alpha)\, \bar{\zeta}_\rho]\, \bar{w} \end{aligned} \tag{19}$$

One should note that, with $\bar{\varepsilon} = \varepsilon + 2\alpha$, $\beta_{\bar{u}}$ is precisely the β-function of the LR theory, Eq. (9). Thus, we identify the second group as the hybrid partners of the LR fixed points. The equation for $\beta_{\bar{u}}$ has a trivial solution $\bar{u}^* = 0$ and a nontrivial one, with $\bar{u}^* \neq 0$. Seeking the corresponding values for $\bar{w}^*$, we obtain a "Gaussian" fixed point (v) $(0, 0, \infty)$ and a nontrivial one (vi) with $\bar{u}^* = 32(\varepsilon + 2\alpha)/5 + O(\varepsilon^2)$, $\bar{w}^* = 0$ and $v^* = (4\alpha - 3\varepsilon)/[3(\varepsilon + 2\alpha)] + 2\alpha/3 + O(\varepsilon^2)$. Note that we have expanded $\bar{u}^*$ in both ε and α, keeping only terms to first order in either, since both are assumed to be of the same order. v^* is $O(1)$ which should not be disturbing since v is not a perturbative coupling. Since $\bar{u}^* \neq 0$ here, we find $\bar{\zeta}_\rho^* = -2(\varepsilon + 2\alpha)/(5 - 2\alpha)$ to all orders.

Turning to the stability of these fixed points, it is straightforward to determine that the Gaussian line is stable for $d > 2(1+\alpha)$, i.e., above the upper critical dimension of the LR theory. For $d < 2(1+\alpha)$, we find a more complex situation: for $\alpha < \alpha_1$, fixed point (vi) has two unstable directions. One of these becomes stable above α_1. For $\alpha > \alpha_2$, this fixed point is globally stable.

It is natural, of course, to seek fixed points where neither $\bar{u}^*$ nor $\bar{w}^*$ vanish. This attempt gives us a pair of equalities, valid to all orders:

$$\begin{aligned} 0 &= (\varepsilon + 2\alpha) + (\tfrac{5}{2} - \alpha)\, \bar{\zeta}_\rho^* \\ 0 &= 2\alpha + (1-\alpha)\, \bar{\zeta}_\rho^* \end{aligned} \tag{20}$$

which do *not* result in a unique equation for such a fixed point. Instead, they select a specific value of α, namely $\alpha = \alpha_1 = \varepsilon/(3+\varepsilon)$, where a fixed *line* (vii) exists, parameterized by $\bar{w}$: $(32\varepsilon/[3(3\bar{w}+1)], \bar{w}, (3\bar{w}-1)(1-\bar{w})/(3\bar{w}^2+2\bar{w}+3))$. This line mediates the stability loss of fixed point (i).

The crossover scenario between the SR and the true LR theory can now be summarized. Let us fix d just below 2 and increase α starting from zero. For sufficiently small $0 \leqslant \alpha < \alpha_1$, the RG flow is dominated by the SR behavior. The SR fixed points (i) and (ii) are stable, within their respective domains of attraction. The separatrix forms a surface which cuts the $v=0$ plane at $\bar{w} = 1/3 + O(\varepsilon)$ and then bends over to larger values of $\bar{w}$ as v increases. The unstable LR fixed point (vi) is found on the separatrix, in the unphysical region $v<0$. When α reaches α_1, the correlated noise begins to make its presence felt. Specifically, for $\alpha_1 < \alpha < \alpha_2$, the FDT violating fixed point (ii) is the only globally stable fixed point. The FDT restoring fixed point (i), while still stable *within* the $v=0$ plane, has become unstable to small perturbations *out* of that plane. Thus, a flow line starting near (i), with a small $v>0$ component, will first flow out into the half-space $v>0$ and then bend *back* towards $v=0$, flowing into the FDT violating fixed point (ii). The *sign* of v remains invariant under the flow. The LR fixed point (vi) is still unphysical, but has moved closer to the $v=0$ plane. Finally, at $\alpha = \alpha_2$, (vi) merges with (ii) and moves out into the positive v region as α increases beyond α_2. The LR fixed point (vi) is now the only stable one, and the global RG flow is dominated by the LR theory. A different view of the SR-LR crossover is presented in Fig. 1 which shows the location of the stability boundaries as functions of α and d.

It is interesting to note the two different scenarios which control the stability loss of fixed points here. Fixed points (ii) and (vi) exchange their stability by merging, similar to the stability exchange between the Gaussian

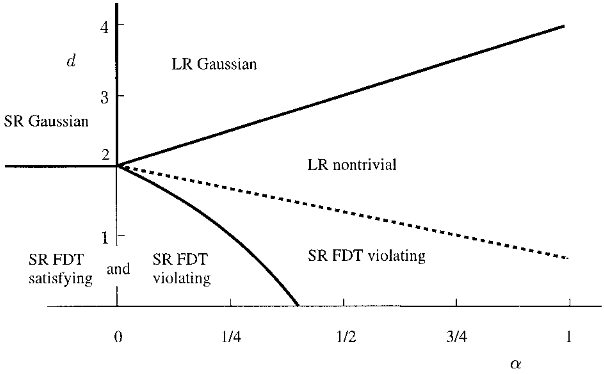

Fig. 1. Stability boundaries of different fixed points as functions of d and α. The heavy solid lines denote boundaries whose location is known to all orders in perturbation theory.

and the Wilson–Fisher[20] fixed points in ϕ^4-theory when d drops below 4. In contrast, fixed point (i) loses its stability, at $\alpha = \alpha_1$, by generating a *fixed line*. This fixed line has two key properties: first, it connects fixed points (i), (iii) and (vi); second, it lies on the surface forming the separatrix between fixed points (i) and (ii). This surface bends back towards the $v = 0$ plane, as α increases, to the extent that it *reaches* fixed point (i) at $\alpha = \alpha_1$. Thus, fixed point (i) loses one of its three stable directions when being absorbed by the separatrix; moreover, this direction is now spanned by the fixed line: this allows the flow to change sign when α crosses α_1.

The crossover which we observe in the flow diagrams is also reflected by the exponents. As an example, we consider the scaling form of the LR structure factor, Eq. (10). We recall that its scaling behavior is determined by the exponent $\Delta_{LR} = \bar{\varepsilon}/(5 - 2\alpha)$. The hybrid fixed point (vi), on the other hand, generates an exponent $\Delta \equiv -\bar{\zeta}_\rho^*/2 = (\varepsilon + 2\alpha)/(5 - 2\alpha)$. These two exponents clearly match, since $\bar{\varepsilon} = \varepsilon + 2\alpha$. As α decreases, we reach the stability boundary $\alpha_2 = 3\varepsilon/4 + O(\varepsilon^2)$, where (vi) and (ii) exchange stability. Here, the exponent associated with (ii) is $\Delta_{SR} = -\zeta_\rho^{SR*}/2 = \varepsilon/2 + O(\varepsilon^2)$, to be compared with the corresponding value near (vi), $\Delta = (\varepsilon + 2\alpha_2)/(5 - 2\alpha_2)$. In this way, continuity is ensured. However, a discontinuous change of exponents may occur as α decreases below α_1: above α_1 all positive $(\bar{u}, \bar{w}, v)$ fall into the domain of attraction of fixed point (ii), with $\Delta_{SR} = \varepsilon/2 + O(\varepsilon^2)$; in contrast, below as, some of these will be attracted towards fixed point (i) where $\Delta_{SR} = \varepsilon/3$. For these theories, the strong anisotropy exponent Δ will change *discontinuously* upon crossing the stability boundary between fixed points (i) and (ii). Note, however, that $\Delta_{SR} = \varepsilon/3$ coincides with $\Delta_{LR} = \bar{\varepsilon}/(5 - 2\alpha)$ at the line $\alpha = \alpha_1 = \varepsilon/(3+\varepsilon)$ to all orders in ε. Nevertheless, even though the exponents may undergo discontinuities, both the hybrid and the SR structure factors scale according to Eq. (7). Thus, the scaling *forms* remain unchanged.

IV. CONCLUSIONS

Using field theoretic methods, we have analyzed the RG flow for a model of biased diffusion subject to a noise term, parameterized by its momentum dependence, $q_{\|}^{2(1-\alpha)}$, with spatially long-ranged correlations. One limit, $\alpha = 0$, corresponds to a "short-range" model with purely local, conserved noise, and the other, $\alpha = 1$, models biased diffusion with a non-conserved noise. The crossover from $\alpha < 1$ to $\alpha = 1$ presents no difficulties; however, the opposite limit, $\alpha \to 0$, is quite subtle. Here, the full crossover is observed only if we interpose a hybrid model with $\alpha = 0(\varepsilon)$, between the SR ($\alpha = 0$) and the LR (α finite) theories. This hybrid contains the key elements of both, SR and LR, Langevin equations. It possesses a number of fixed points, including the equivalents of the SR and the LR models, so that the crossover can be understood in terms of a stability exchange between different fixed points. Clearly, considering α on the scale of ε enables us to resolve this "fine structure." Just below two dimensions, the scenario is the following: for α below a lower stability limit α_1, the theory is controlled by two stable SR fixed points, one FDT restoring and the other FDT violating, each with its own basin of attraction. Stated differently, we are in a "weak noise" regime, for $d < d_{\mathrm{LR}} \equiv 2 - 3\alpha/(1-\alpha)$, where the long-ranged noise does not significantly modify the universal behavior of the system. As α increases beyond α_1 but remains below an upper stability limit α_2, the SR FDT restoring fixed point becomes unstable, leaving the SR FDT violating fixed point in control of the flow. This stability exchange is mediated by a fixed point line. Finally, above α_2, the SR FDT violating fixed point also destabilizes and the nontrivial LR fixed point becomes globally stable. We note, in conclusion, that a similar mechanism, namely stability exchange through a fixed point line, has previously been observed in the Sine–Gordon model: there, two fixed points are stable below $d = 2$, corresponding to the high- and low-temperature phases respectively. As d increases beyond 2, the high-temperature fixed point loses its stability, via a similar fixed point line.[21]

Unfortunately, most of the nontrivial crossover phenomena discussed here are confined to dimensions $1 < d < 2$. Above $d = 2$, the long-range noise dominates, either via its nontrivial or its Gaussian fixed point. In one dimension, on the other hand, there is no transverse subspace so that the SR FDT violating fixed point cannot be accessed. Consequently, the scenario described above must change significantly. It is conceivable that only the lower stability limit might survive here or that the two stability limits merge, leaving us with a stability exchange between the FDT restoring SR and the LR fixed point. Since, at $\alpha = \alpha_1$, $\Delta_{SR} = \Delta_{LR} = \varepsilon/3$ to all orders, this may be a reasonable conjecture. Moreover, in $d = 1$, we have $\alpha_1 = 1/4$ and $\Delta = 1/3$. These values agree with the corresponding results[13, 14] for the one-dimensional KPZ equation with correlated noise where only one stability limit is observed.

Nevertheless, our analysis plays the role of a pilot study for a number of other interesting problems. Clearly, one might consider an interacting theory subject to an external bias[19] and a long-range noise term.[16] Here, $d_c = 5$, so that physical dimensions are more accessible, and comparisons with Monte Carlo simulations can be made. Other questions of interest concern noise terms with different spatial correlations, or nontrivial correlations in time. For equilibrium systems, the existence of an underlying Hamiltonian ensures that these correlations have no effect on static properties. For non-equilibrium steady states, however, the static behavior is generically inseparable from the dynamics. Studies of anomalous noise correlations in non-equilibrium systems may help to unravel the nature of this coupling.

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